A Special case of Zykov's theorem and the shifting method

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PRIMES Circle

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Introduction to Extremal Graph Theory

Extremal graph theory focuses on finding the maximum and minimum possible numbers of occurrences of certain patterns in graphs under various conditions. Study of extremal graph theory began in early 20^{th} century with a theorem of Mantel

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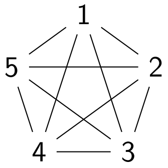
Notation

K_r (clique of size r) is a set of r vertices, such that each pair is connected by an edge.



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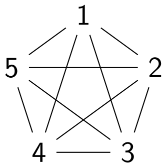

Theorem (Turán (1941))

Let $r \geq 3$ and n be positive integers. Any n-vertex graph which does not contain a K_r has at most $\frac{(r-2)n^2}{2(r-1)}$ edges.

Problem Statement

Theorem (Zykov 1949)

Let $l > k \geq 2$ be integers. Any n-vertex graph without any K_l has at most

$$\frac{n^2}{(l-1)^2}\binom{l-1}{k}$$

copies of K_k.

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We will prove this theorem for $k = 3$, $l = 5$ (the general case is similar).

Question

Prove that a K_5 free graph on n vertices has $\leq \frac{n^3}{16}$ triangles.

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Let $t(u)$ be the number of triangles containing vertex u, and $t(uv)$ be the number of triangles containing edge uv. Let also $t(G)$ be the number of triangles in a graph G.

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Let $G = (V, E)$ be such that

- G has no K_5's and the most triangles
- Any edge not contained in any triangles can be removed without changing the number of triangles
- Thus, $t(uv) \geq 1$ for all $uv \in E$

Symmetrization

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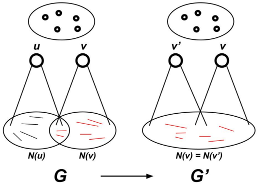

Consider vertices u and v such that $uv \notin E$ and $t(u) < t(v)$. Then remove u and replace it with a new vertex v', such that $N(v') = N(v)$, to create G'.

$$t(G') = t(G) - t(u) + t(v) > t(G)\,.$$

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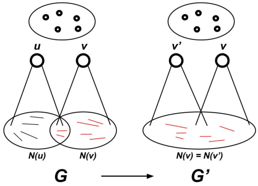

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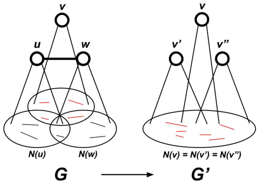
If G' contains a K_5, then $N(v)$ contains a K_4, so G contained a K_5. Thus G' is K_5-free and has more triangles than G, which is a contradiction.

Symmetrization

Lemma

If $uv, vw \notin E$ then $uw \notin E$.

Consider vertices u, v, w such that $uv, vw \notin E$. Replace u with v' and w with v'', such that $N(v') = N(v'') = N(v)$, to create G'.



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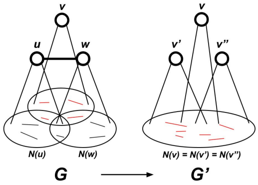
$$\begin{aligned} t(G') &= t(G) - t(u) - t(w) + t(uw) + 2t(v) \\ &= t(G) + t(uw) \\ &> t(G)\,. \end{aligned}$$



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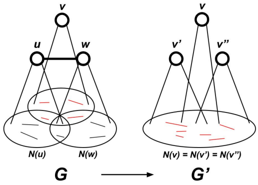

$$t(G') = t(G) - t(u) - t(w) + t(uw) + 2t(v)$$
$$= t(G) + t(uw)$$
$$> t(G).$$

Similarly as before, G' must be K_5-free, and it has more triangles than G, which is a contradiction.

Fact

If an graph $G = (V, E)$ is such that for any $uv, vw \notin E$ we have $uw \notin E$, then G must be a complete multipartite graph.

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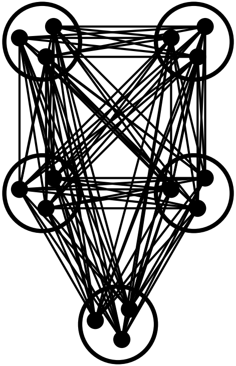

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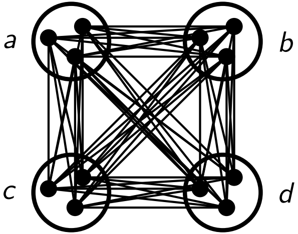
By previous lemma, G is a complete multipartite graph. G is K_5 free, so it is a 4-partite graph. Let a, b, c, d be the numbers of vertices in 4 parts.



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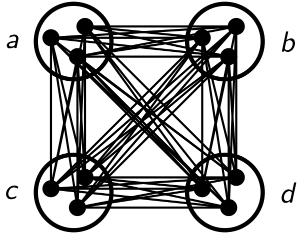

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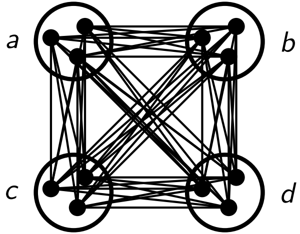

Triangle count $= abc + abd + acd + bcd$

K_5-free graph on n vertices has $\leq \frac{n^3}{16}$ triangles.

$$\Updownarrow$$

For any $a, b, c, d \geq 0$

$$abc + abd + acd + bcd \leq \frac{(a+b+c+d)^3}{16}.$$

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If $a = b = c = d$, the equality holds. We can use the shifting method to achieve this condition. Let $M = \frac{a+b+c+d}{4}$ be the mean of a, b, c, d.

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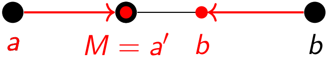
2. $(a, b) \longrightarrow (a' = a + x,\ b' = b - x)$, where $x = \min((M - a), (b - M))$.



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3. After shifting, the LHS increased while the RHS stayed the same.

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\begin{aligned}
LHS &= (a+x)(b-x)c + (a+x)(b-x)d + (a+x)cd + (b-x)cd \\
&= abc + abd + acd + bcd + (-ax + bx - x^2)(c+d) \\
&= abc + abd + acd + bcd + (b-a-x)(c+d)x \\
&> abc + abd + acd + bcd
\end{aligned}
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RHS = \frac{((a+x) + (b-x) + c + d)^3}{16} = \frac{(a+b+c+d)^3}{16}.
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$$RHS = \frac{((a+x)+(b-x)+c+d)^3}{16} = \frac{(a+b+c+d)^3}{16}.$$

4. Keep shifting until we reach $a = b = c = d$, when LHS = RHS. Because LHS increased and RHS did not, thus at the beginning LHS $\leq$ RHS.

$$abc + abd + acd + bcd \le \frac{(a+b+c+d)^3}{16}$$

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